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Expansion into Partial Fractions

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Calculating Coefficients in the Partial Fraction Decomposition

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There are several methods of calculating the unknown coefficients appearing in the partial fraction expansion of a fraction of two polynomials. This paper shows how to derive a method based on standard results from the theory of complex variables, namely, the Cauchy-Goursat theorem and the Cauchy integral formula.

Assume $f(z) = \dfrac{P(z)}{Q(z)}$ with $P(z)$ and $Q(z)$ polynomials of degrees m and n respectively and with m < n. We first consider the case when $Q(z)$ has a real root of multiplicity r.

Therefore, we have the expansion

$$\frac{P(z)}{Q(z)} = \frac{L_1}{(z-a)^r} + \frac{L_2}{(z-a)^{r-1}} + ... + \frac{L_r}{z-a} + H(z) \tag{1}$$

where the L_k's, k=1 ,2,....r are unknown constants and H(z) is the remaining part of the expansion.

Assume $Q(z)= z^n + b_{n-1}z^{n-1} + ... + b_1 z + b_0$ (there is no loss of generality assuming the coefficient of z^n as one). From (1) we obtain

$$P(z) = T(z)\left[L_1 + L_2(z-a) + ... + L_r(z-a)^{r-1}\right] + (z-a)^r \,\mathrm{N(z)} \tag{2}$$

where

$$N(z) = T(z)H(z) \tag{3}$$

$$T(z) = \frac{Q(z)}{(z-a)^r} \tag{4}$$

and T does not vanish at $z = a$. If we let $z = a$ in (2) we obtain $P(a) = T(a)L_1$ and

therefore, $L_1 = \dfrac{P(a)}{T(a)}$ (5)

Now assume that L_1 through L_k have been defined. If we take the terms in (2) involving L_1 through L_k to the left side and divide both sides by $(z-a)^{k+1}$, equation (2) becomes

$$\frac{P(z) - T(z)\left[L_1 + L_2(z-a) + ... + L_k(z-a)^{k-1}\right]}{(z-a)^{k+1}} = T(z)\left[\frac{L_{k+1}}{z-a} + L_{k+2} + ... + L_r(z-a)^{r-k-2}\right] + (z-a)^{r-k-1} N(z) \tag{6}$$

Now assume C is a simple closed curve containing $z = a$ in its interior, but no other root of Q(z) lies in the interior of C. Integrating both sides of equation (6) along C yields

$$\oint_C \frac{P(z)-T(z)\left[L_1+L_2(z-a)+.....+L_k(z-a)^{k-1}\right]}{(z-a)^{k+1}}dz = \oint_C \frac{T(z)L_{k+1}}{z-a}dz + \oint \mathrm{T(z)}\left[\mathrm{L}_{k+2}+...+L_r(z-a)^{r-k-2}\right]dz + \oint_C (z-a)^{r-k-1}N(z)dz \tag{7}$$

Since the integrands of the second and third integrals on the right side of (7) are analytic functions on and inside the curve C, these integrals are zero by the Cauchy-Goursat theorem.

Applying the Cauchy integral formula

$$g(a) = \frac{1}{2\pi i}\oint_C \frac{g(z)}{z-a}dz \tag{8}$$

to the first integral on the right hand side and the deriviative form of the Cauchy integral formula

$$g^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{g(z)}{(z-a)^{n+1}}dz \tag{9}$$

to the left hand side, it is easy to see that

$$L_{k+1} = \frac{P^{(k)}(a) - \dfrac{d^k}{dz^k}[T(z)\sum_{m=0}^{k-1}L_{m+1}(z-a)^m]_{z=a}}{k!T(a)} \tag{10}$$

Next, applying Leibnitz's rule for the derivative of the product of two functions

$$(uv)^{(n)} = \sum_{h=0}^{n} C_n^h u^{(n-h)} v^{(h)} \tag{11}$$

to the term with minus sign of the numerator in (10) gives

$$\frac{d^k}{dz^k}[T(z)\sum_{m=0}^{k-1}L_{m+1}(z-a)^m]_{z=a} = \sum_{h=0}^{k}\{C_k^h \frac{d^{k-h}}{dz^{k-h}}[\sum_{m=0}^{k-1}L_{m+1}(z-a)^m]_{z=a}T^{(h)}(a)\} \tag{12}$$

After some simplifications we obtain

$$\frac{d^{k-h}}{dz^{k-h}}[\sum_{m=0}^{k-1}L_{m+1}(z-a)^m]_{z=a} = (k-h)!L_{k-h+1} \tag{13}$$

using

$$C_k^h = \frac{k!}{h!(k-h)!} \tag{14}$$

Hence, equation (12) is now

$$\frac{d^k}{dz^k}[T(z)\sum_{m=0}^{k-1} L_{m+1}(z-a)^m = \sum_{h=1}^{k} \frac{k!}{h!} L_{k-h+1} T^{(h)}(a)$$

and therefore equation (10) becomes

$$L_{k+1} = \frac{P^{(k)}(a) - \sum_{h=1}^{k} \frac{k!}{h!} T^{(h)}(a) L_{k-h+1}}{k! T(a)} \tag{15a}$$

Expanding, we have

$$L_{k+1} = \frac{P^{(k)}(a) - [T^{(k)}(a)L_1 + T^{(k-1)}(a)kL_2 + T^{(k-2)}(a)k(k-1)L_3 + ... + T^{'}(a)k!L_k]}{k!T(a)} \tag{15b}$$

k=1, 2, ..., r-1

where the superscripts mean derivatives. For example, if k=1 we have

$L_2 = \dfrac{P'(a) - T'(a)L_1}{T(a)}$. Recall that $L_1 = \dfrac{P(a)}{T(a)}$ has already been determined.

If k=2, equation (15a) gives

$$L_3 = \frac{P^{"}(a) - [T^{"}(a)L_1 + 2T'(a)L_2]}{2T(a)}$$

Since we already know L_1 and L_2, we can calculate L_3. Continuing in this manner allows us to find the remaining coefficients L_4 through L_r .

II.-Case of irreducible quadratic forms.

In the case where Q(z) has an irreducible quadratic term in its factoriztion, the partial fraction decomposition would appear as we write the expression z^2 +bz +c which contains the complex roots a and a^*.

$$\frac{P(z)}{Q(z)} = \frac{A_1 z + B_1}{(z^2 + bz + c)^{\nu}} + \frac{A_2 z + B_2}{(z^2 + bz + c)^{\nu-1}} + ... + \frac{A_{\nu} z + B_{\nu}}{z^2 + bz + c} + J(z) \tag{16}$$

Where the A_k 's , B_k 's, b and c are real numbers and J(z) is the remaining terms in the decomposition. Furthermore assume the irreducible term can be factored as

$$z^2 + bz + c = (z-a)(z-a^*) \tag{17}$$

where a is a complex number and a^* is its complex conjugate. Equation 16 can now be rewritten as

$$\begin{aligned} P(z) &= U(z)[A_1 z + B_1 + (z-a)(z-a^*)(A_2 z + B_2) + (z-a)^2 (z-a^*)^2 (A_3 z + B_3) + \\ &\dots + (z-a)^{\nu-1}(z-a^*)^{\nu-1}(A_\nu z + B_\nu)] + (z-a)^\nu G(z) \end{aligned} \tag{18}$$

with

$$U(z) = \frac{Q(z)}{(z^2+bz+c)^\nu} \qquad \text{and} \tag{19}$$

$$G(z) = (z-a^*)^\nu U(z) J(z) \tag{20}$$

Letting $z = a$ in (18), we have

$$P(a) = U(a)(A_1 a + B_1) \quad \text{and} \quad \mathrm{A}_1 a + \mathrm{B}_1 = \frac{\mathrm{P}(a)}{\mathrm{U}(a)} \tag{21}$$

Since the right side is a complex number as well, equating the corresponding real and imaginary parts we get two equations which can be used to determine A_1 and B_1.

Now if we rearrange (18) as in the previous case, we get

$$\begin{aligned} &\frac{P(z) - U(z)\sum_{k=0}^{m-1}(z-a)^k (z-a)^k (A_{k+1} z + B_{k+1})}{(z-a)^{m+1}} = U(z)[\frac{(z-a^*)(A_{m+1}z + B_{m+1})}{z-a} + \\ &(\mathrm{z}-\mathrm{a}^*)(A_{m+2} z + B_{m+2}) + \dots + (\mathrm{z}-\mathrm{a}^*)(A_\nu z + B_\nu)^{\nu\text{-m-2}}] + (\mathrm{z}-\mathrm{a})^{\nu\text{-m-1}} G(z) \end{aligned} \tag{22}$$

Again, when we integrate around a closed curve C whose interior contains $z = a$, equation (22) becomes

$$\frac{P^{(m)}(a) - \frac{d^m}{dz^m}[U(z)\sum_{k=0}^{m-1}(z-a)^k (z-a^*)^k (A_{k+1} z + B_{k+1})]_{z=a}}{m!} = U(a)(a-a^*)(A_{m+1} a + B_{m+1}) \tag{23}$$

Therefore,

$$A_{m+1} a + B_{m+1} = \frac{P^{(m)}(a) - \{U(a)\sum_{k=0}^{m-1}(z-a)^k (z-a^*)^k (A_{k+1} z + B_{k+1})\}^{(m)}|_{z=a}}{m!(a-a^*)U(a)} \tag{24}$$

If we define

$$F(z) = U(z)\sum (z-a)^k (z-a^*)^k (A_{k+1} z + B_{k+1}) \tag{25}$$

and apply Leibnitz's rule as before to $F^{(m)}(a)$, we obtain, after some long calculations,

$$F^{(m)}(a) = \sum_{h=0}^{m} \{C_m^h U^{(m-h)}(a) \sum_{k=0,\leq h}^{m-1} \frac{h!}{(h-k)!} [(z-a^*)^k \ (\mathrm{A}_{k+1} z + \mathrm{B}_{k+1})]^{(h-k)} \ |_{z=a}\} \tag{26}$$

Therefore equation (24) becomes :

$$A_{m+1}a + B_{m+1} = \frac{P^{(m)}(a) - F^{(m)}(a)}{m!(a-a^*)^m U(a)} \tag{27}$$

m=1, 2, 3, ..., ν-1.

Since we know A_1 and B_1, the other coefficients can be found.

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